Closing Tues: 2.8
Closing Thurs: 3.1-2
Closing Fri: 3.3

Exam 1 is Tues, Oct. 17 in your normal quiz section. Covers: 2.1-3,2.5-8, 3.1-3.

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(cf(x)) = cf'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}$

Entry Task: Find the derivatives of

$a)\ g(x) = \frac{x^3}{2} - \frac{3}{\sqrt{x^5}}$

$b)\ f(x) = \frac{20}{3}x^3 - \frac{7x^2}{2} - 6x + 90$

$c)$ Find all x at which $y = f(x)$ has a horizontal tangent.

(a) $g(x) = \frac{1}{2}x^3 - 3x^{-5/2}$

$g'(x) = \frac{1}{2}3x^2 - 3\left(-\frac{5}{2}\right)x^{-7/2}$

$g'(x) = \frac{3}{2}x^2 + \frac{15}{2}x^{-7/2}$

(b) $f(x) = \frac{20}{3}x^3 - \frac{7}{2}x^2 - 6x + 90$

$f'(x) = 20x^2 - 7x - 6$

(c) WANT ALL x WHERE THE SLOPE OF THE TANGENT TO $f(x)$ IS ZERO!

$20x^2 - 7x - 6 \overset{?}{=} 0$

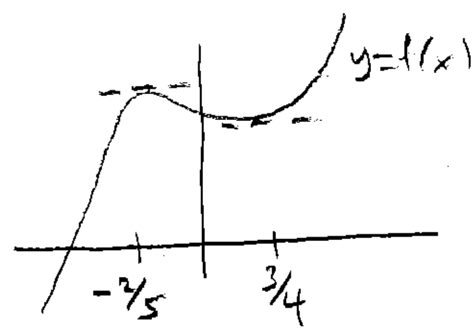
or

$(4x-3)(5x+2) = 0$

$x = 3/4, \quad x = -2/5$

$x = \frac{7 \pm \sqrt{49 - 4(20)(-6)}}{2(20)}$

$x = \frac{7 \pm 23}{40} = \begin{cases} \frac{30}{40} = 3/4 \\ \text{or} \\ \frac{-16}{40} = -2/5 \end{cases}$



Application Notes:

1. $f'(a) =$ "slope of tangent to f(x) at *a*"
2. *Tangent Line Equation*:

$$y = f'(a)(x-a) + f(a)$$

3. $-\frac{1}{f'(a)} =$ "slope of *normal* to f(x) at *a*"
4. *Normal Line Equation*:

$$y = -\frac{1}{f'(a)}(x-a) + f(a)$$

Example: Let $f(x) = x^2 + 3$.

a) Find $f'(x)$

b) Find the equations for the tangent and normal lines at x = 2.

c) Find all points on y = f(x) at which the normal line would also pass through (0,10)

(a) $f'(x) = 2x$

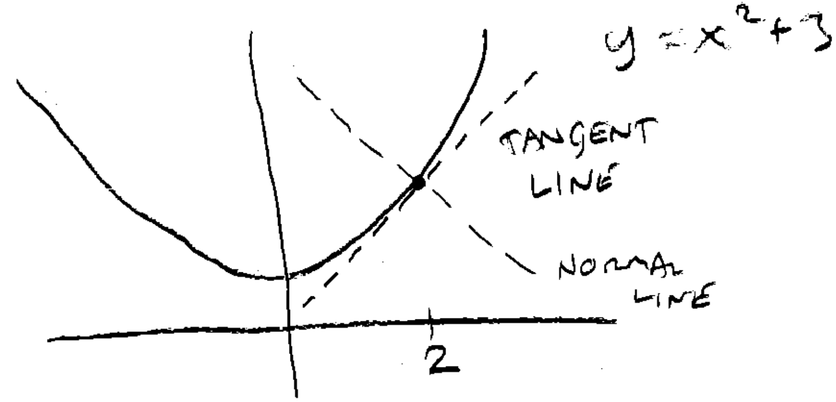
(b) $f(2) = (2)^2 + 3 = 7$

$f'(2) = 2(2) = 4 =$ slope of tangent

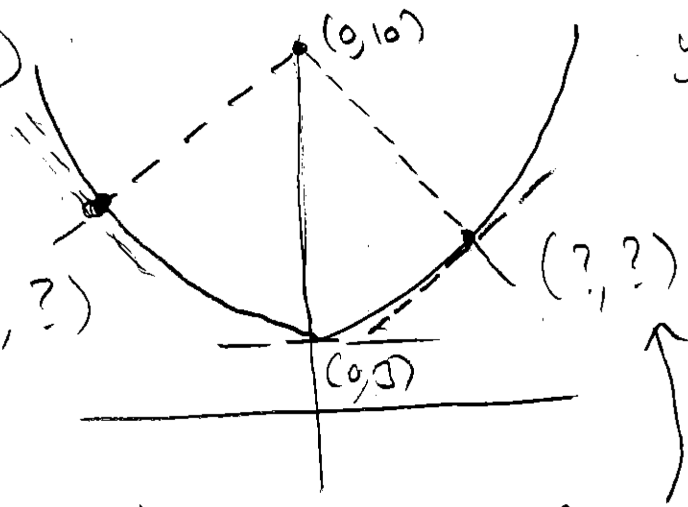
$-\frac{1}{f'(2)} = -\frac{1}{4} =$ slope of normal

$y = 4(x-2) + 7$ ← TANGENT LINE

$y = -\frac{1}{4}(x-2) + 7$ ← NORMAL LINE



(c)



$y = x^2 + 3$

NOTE WE CAN SEE THAT (0,3) IS ONE SOL'N

step 1: DRAW & LABEL (a, b)

step 2: WRITE DOWN ALL KNOWN FACTS

$\boxed{\text{I}}$ (a,b) IS ON THE CURVE $\Rightarrow b = a^2 + 3$

$\boxed{\text{II}}$ THE DESIRED LINE GOES THROUGH (0,10) AND (a,b) $\Rightarrow$ DESIRED SLOPE $= \dfrac{b-10}{a-0}$

$\boxed{\text{III}}$ "TANGENT SLOPE AT (a,b)" $= 2a$

"NORMAL SLOPE AT (a,b)" $= -\dfrac{1}{2a}$

WANT THESE THE SAME

step 3: COMBINE & SOLVE

(i) $b = a^2 + 3$, and

(ii) $-\dfrac{1}{2a} \stackrel{?}{=} \dfrac{b-10}{a} \Rightarrow -\frac{1}{2} = b - 10$

(ii) $\Rightarrow b = 10 - \frac{1}{2} = \dfrac{19}{2}$

(ii) & (i) $\Rightarrow \dfrac{19}{2} = a^2 + 3$

$$\frac{19}{2} - 3 = a^2$$

$$\frac{13}{2} = a^2$$

$$a = \pm\sqrt{\frac{13}{2}}$$

$(a,b) = \left(-\sqrt{\frac{13}{2}}, \frac{19}{2}\right)$ AND

$\left(\sqrt{\frac{13}{2}}, \frac{19}{2}\right)$ AND

$(0,3)$

CHECK!

NORMAL LINE

$$y = -\frac{1}{2\sqrt{\frac{13}{2}}}\left(x - \sqrt{\frac{13}{2}}\right) + \frac{19}{2}$$

if you plug in $x=0$ do you get 10?

$$y = -\frac{1}{2\sqrt{\frac{13}{2}}}\left(0 - \sqrt{\frac{13}{2}}\right) + \frac{19}{2} = 10 \checkmark$$

Finishing 3.1 (Another Rule)

5. $\dfrac{d}{dx}(e^x) = e^x$ and $\dfrac{d}{dx}(a^x) = a^x \ln(a)$

BY NUMERICAL EXPERIMENTATION

WE APPROXIMATE,

$$\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.693147$$

$$\lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.098612$$

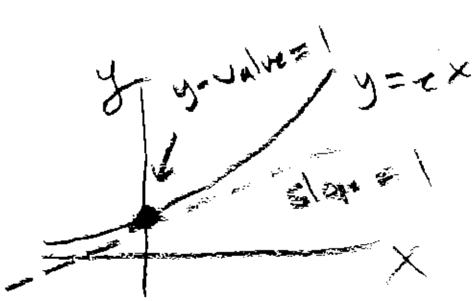
$$\lim_{h\to 0} \frac{2.7182^h - 1}{h} \approx 0.999970$$

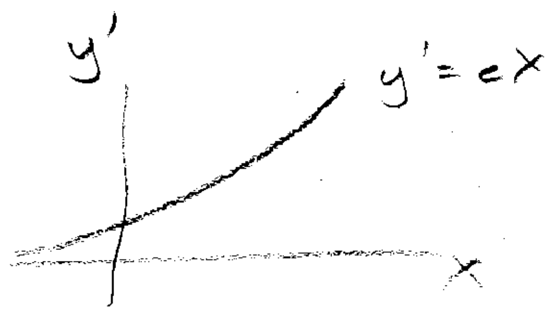
There is a number, around 2.71828182...,

WHERE THIS LIMIT IS 1.

WE CALL THIS NUMBER, $\boxed{e \approx 2.71828182\ldots}$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$





"Proof"

5. Exponential Function Rule:

For $f(x) = a^x$,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

Note: $\ln(a) = \lim_{h\to 0} \dfrac{a^h - 1}{h}$

ASIDE

$0.693147 \approx \ln(2)$

$1.098612 \approx \ln(3)$

$0.999970 \approx \ln(2.7182)$

Find the derivative

1. $y = 5x + 3x^3 + 7e^x$

$$\boxed{y' = 5 + 9x^2 + 7e^x}$$

2. $y = \frac{4(2)^x}{3} - 11 + \frac{8\sqrt[3]{x^2}}{5}$

$$y = \frac{4}{3}(2)^x - 11 + \frac{8}{5}x^{2/3}$$

$$y' = \frac{4}{3}2^x \ln(2) - 0 + \frac{8}{5}\cdot\frac{2}{3}x^{-1/3}$$

$$\boxed{y' = \frac{4}{3}2^x \ln(2) + \frac{16}{15}x^{-1/3}}$$

3.2 Product and Quotient Rules

6. $\frac{d}{dx}\left(\overbrace{f(x)}^{F}\overbrace{g(x)}^{S}\right) = f(x)g'(x) + f'(x)g(x)$

7. $\frac{d}{dx}\left(\frac{f(x) \leftarrow N}{g(x) \leftarrow D}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Examples: Find y'

$$a)\ y = \overset{f(x)}{x^3}\overset{g(x)}{e^x}$$

$$\frac{dy}{dx} = x^3(e^x) + (3x^2)e^x$$

$$= x^3e^x + 3x^2e^x$$

$$= x^2e^x(x+3)$$

$$b)\ y = \frac{2x^4 \leftarrow f(x)}{x^2 - 3 \leftarrow g(x)}$$

$$\frac{dy}{dx} = \frac{(x^2-3)(8x^3) - 2x^4(2x)}{(x^2-3)^2}$$

$$= \frac{8x^3(x^2-3) - 4x^5}{(x^2-3)^2}$$

$$= \frac{4x^3(2(x^2-3) - x^2)}{(x^2-3)^2}$$

$2x^2 - 6 - x^2$

$x^2 - 6$

$$= \frac{4x^3(x^2-6)}{(x^2-3)^2}$$

6. Product Rule Proof:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x+h)g(x)+f(x+h)g(x)}-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

You do: Find $\frac{dy}{dx}$

$a)\ y=x^6e^x-\frac{5}{\sqrt{x^3}}$

$y = x^6 e^x - 5x^{3/2}$

F S

$y' = x^6e^x + 6x^5e^x \quad -5\tfrac{3}{2}x^{1/2}$

$$\boxed{y' = x^5e^x(x+6) - \tfrac{15}{2}\sqrt{x}}$$

$b)\ y=(x^3+1)^2+\frac{x^5}{1+e^x}$

$y = x^6+2x^3+1 + \frac{x^5}{1+e^x}$

$y' = 6x^5+6x^2+0+\frac{(1+e^x)(5x^4)-x^5(e^x)}{(1+e^x)^2}$

$$\boxed{y' = 6x^5+6x^2+\frac{x^4(5+5e^x-xe^x)}{(1+e^x)^2}}$$

$$c)\ y = \frac{2x^2+1}{x^3e^x} \begin{matrix} \leftarrow N \\ D \end{matrix}$$

$$y' = \frac{x^3e^x(4x) - (2x^2+1)(x^3e^x + 3x^2e^x)}{(x^3e^x)^2}$$

$$y' = \frac{4x^4e^x - x^2e^x(2x^2+1)(x+3)}{x^6e^{2x}}$$

$$y' = \frac{x^2e^x\left(4x^2 - (2x^2+1)(x+3)\right)}{x^6e^{2x}}$$

$$y' = \frac{4x^2 - (2x^2+1)(x+3)}{x^4e^x}$$

$$= \frac{4x^2 - (2x^3+6x^2+x+3)}{x^4e^x}$$

$$= \frac{-2x^3-2x^2-x-3}{4x^4e^x}$$

$$d)\ y = \underbrace{(x^2+3)}_{f(x)}\ \underbrace{\overset{F}{\sqrt{x}}\ \overset{S}{e^x}}_{g(x)}$$

$F = x^{1/2}$
$S = e^x$
$F' = \frac{1}{2}x^{-1/2}$
$= \frac{1}{2\sqrt{x}}$

$$y' = (x^2+3)\left(\sqrt{x}\,e^x + \frac{1}{2\sqrt{x}}e^x\right) + 2x\sqrt{x}\,e^x$$

$$y' = e^x\left[(x^2+3)\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) + 2x\sqrt{x}\right]$$

$$y' = \sqrt{x}\,e^x\left[(x^2+3)\left(1 + \frac{1}{2x}\right) + 2x\right]$$

3.3 Derivatives of Trig Functions

First a review: you will need to know all the following well in Math 124/5/6.

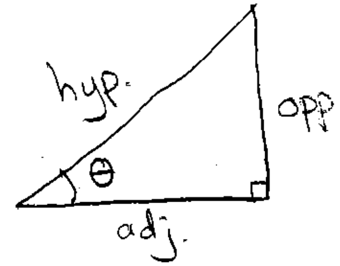
1. Triangle definitions

$\sin(x) = \dfrac{\text{opp}}{\text{hyp}}$	$\cos(x) = \dfrac{\text{adj}}{\text{hyp}}$
$\tan(x) = \dfrac{\text{opp}}{\text{adj}}$	$\cot(x) = \dfrac{\text{adj}}{\text{opp}}$
$\sec(x) = \dfrac{\text{hyp}}{\text{adj}}$	$\csc(x) = \dfrac{\text{hyp}}{\text{opp}}$

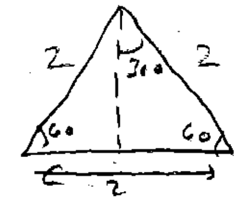
Thus,

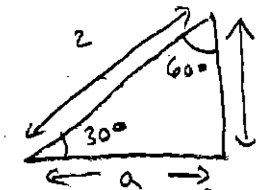
$\sec(x) = \dfrac{1}{\cos(x)}$	$\csc(x) = \dfrac{1}{\sin(x)}$
$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$	$\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

2. **Know what their graphs look like**.
3. **Know their inverses** and how to use them (and how to get more solutions)



Ex)





$\cos(30°) = \frac{\sqrt{3}}{2}$ $\Leftarrow$

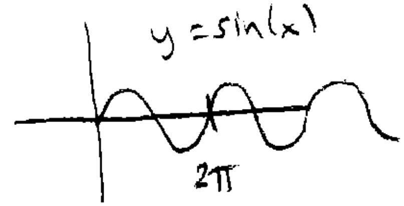
$\sin(30°) = \frac{1}{2}$

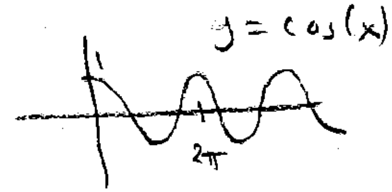
$\tan(30°) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

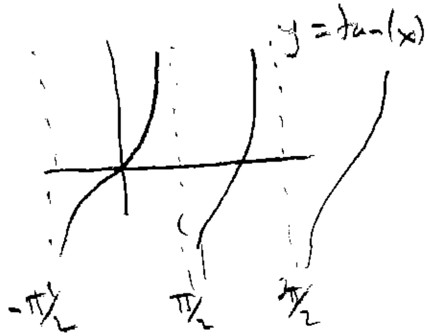
$a^2 + 1^2 = 2^2$

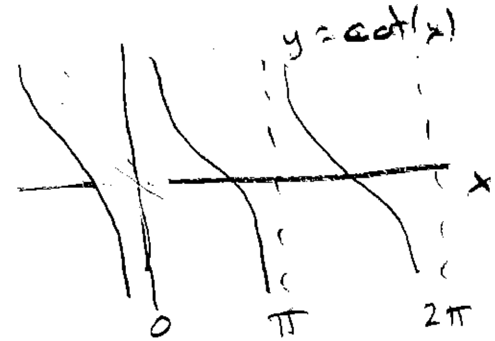
$\Rightarrow a^2 = 3$

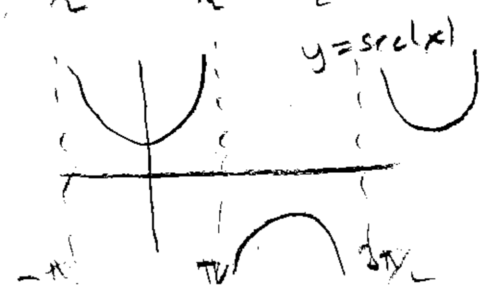
$a = \sqrt{3}$

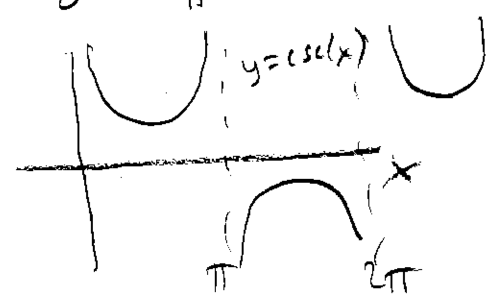












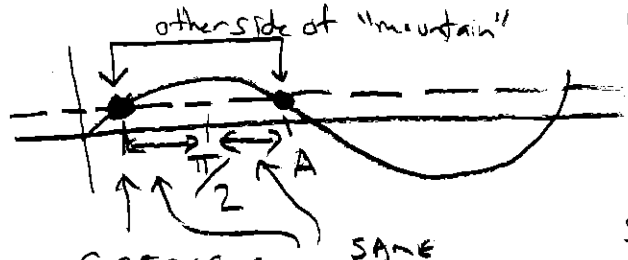
INVERSES

Example To solve $\sin(\theta) = 1/4$

STEP 1 PRINCIPAL SOL'N $\Longleftrightarrow$ $\theta = \sin^{-1}(1/4) \approx 0.2526803$ radians

STEP 2 SYMMETRIC SOL'N $\Longleftrightarrow$ $\theta = \pi - \sin^{-1}(1/4) \approx 2.8889124$ radians

Explanation



SO $A = \frac{\pi}{2} + \left(\underbrace{\frac{\pi}{2} - 0.2526803}\right)$

STEP 3 ALL OTHER SOL'NS

$\theta = \sin^{-1}(1/4) + 2\pi k$ for any integer k

$\theta = \pi - \sin^{-1}(1/4) + 2\pi k$ for any integer k

IF YOU ARE SOLVING $\sin(3t) = 1/4$, THEN THE PROCESS IS THE SAME! JUST REPLACE θ BY $3t$ AT THE <u>END</u> AND DIVIDE BY 3.

$$\Longrightarrow \quad 3t = \sin^{-1}(1/4) + 2\pi k \Rightarrow t = \frac{1}{3}\sin^{-1}(1/4) + \frac{2\pi}{3}k$$

$$3t = \pi - \sin^{-1}(1/4) + 2\pi k \Rightarrow t = \frac{\pi}{3} - \frac{1}{3}\sin^{-1}(1/4) + \frac{2\pi}{3}k$$

4. Know the standard values (unit circle) and circular motion

θ	$\cos\theta$	$\sin\theta$
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1

Examples (do NOT use a calculator)

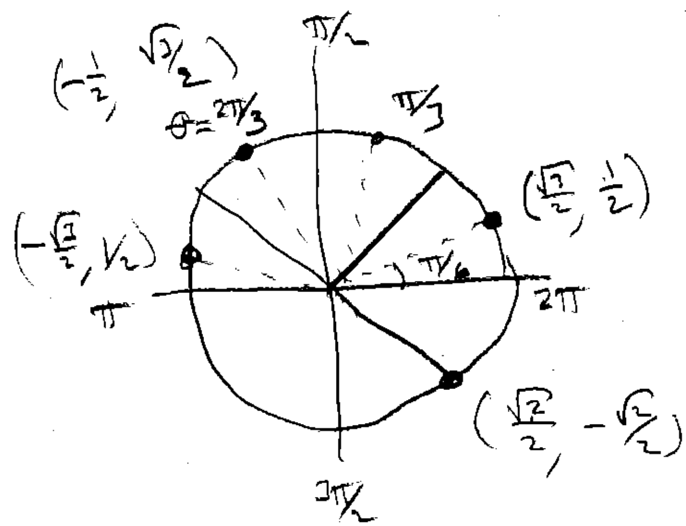
$$\cos\left(\frac{\pi}{6}\right) = \sqrt{3}/2$$

$$\sec\left(-\frac{\pi}{4}\right) = \frac{1}{\cos(-\pi/4)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\tan\left(\frac{2\pi}{3}\right) = \frac{\sin(2\pi/3)}{\cos(2\pi/3)} \quad \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \pi/6$$

$$\tan^{-1}(1) = \pi/4$$



5. Know the main identities.

$$\sin^2(x) + \cos^2(x) = 1$$

$$2\sin(x)\cos(x) = \sin(2x)$$